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Controlling current reversals in synchronized underdamped ratchets

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Abstract
A pair of underdamped ratchets, coupled via a perturbed asymmetric potential, is shown to make a transition to a fully synchronized state wherein stable controlled transport is achieved when the coupling strength exceeds a threshold at which the collective dynamics is attained. This transition to collective transport is connected to a chaos-periodic/quasiperiodic bifurcation in which current reversal is completely eliminated. Based on the Lyapunov stability theory and linear matrix inequalities, we give some necessary and sufficient criteria for stable controlled transport and obtain an exact analytic estimate of the threshold (k_{th}) for the onset of stable controlled current.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The ratchet effect, that is, the possibility of realizing directed transport without net bias in systems driven out of equilibrium, occurs in many natural situations ranging from physical through chemical to biological systems. Current reversal is a particularly intriguing phenomenon that has been central to recent experimental and theoretical investigations of transport based on the ratchet mechanism. Much of the recent research interest in transport problems relates to the physics of molecular motors, where unbiased, noise-induced transport arises away from thermal equilibrium [1–3]. Such Brownian motors, especially 'ratchet' models, have been widely investigated for several reasons: (i) to describe and control the mechanism underlying certain biological processes at both the cellular level as found

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in ion channels and the organ level, for instance muscle contraction [4]; (ii) to construct nanoscale devices for guiding tiny particles aimed at particle separation, smoothing of atomic surfaces during electromigration, and control of the motion of quantized flux vortices in superconductors [2, 5]; and (iii) to explore the potential applications of the rectification power of such devices in the design and operations of high performance and dependable rectifiers, such as arrays of Josephson junction [6], long Josephson junctions [7], asymmetric superconducting quantum interference devices [8] and quantum electronic devices [5]. Some of these possibilities have already been demonstrated in practical applications [9–11].

The large variety of systems that exhibit the ratchet effect can be classified into two basic types. Those in the first class, where the system is driven out of equilibrium by a pulsating force, are called flashing ratchets. The second class is that of rocking (or correlation) ratchets wherein the system is driven by an external unbiased driving force. The vast majority of these models are overdamped, and noise plays a vital role in the transport process. Yet it has been found that deterministic chaos induced by the inertial term in the model equation can equivalently replace the role of noise [12, 13]. For inertial ratchets, moving in asymmetric rocking potentials, the dynamics and transport properties are strongly dependent on the system's parameters as well as on the initial conditions; in particular, the current can suddenly change direction at specific bifurcation points—these issues have been carefully investigated in [12–16] and the effects of noise or disorder have also been reported [17–19].

Theoretical studies of ratchet models have been largely restricted to noninteracting or single-particle systems. However, in reality, the interactions between particles are very important and cannot be ruled out in ratchet systems. For instance, it is well known that molecular motors do not operate as a single particle but congregate in groups that form multimotors—the most prominent example being the actin–myosin system in muscles [20]. Similarly, systems such as microfluidic channels [21], 2DEG nanostructures with strong electron–electron interactions [22] and granulated materials [24, 25], etc represent practical situations where particle–particle interactions are essential. For this reason, the relevance of two or more interacting particles and the effects of their collective dynamics on net transport have attracted attention (e.g. [23, 26–33, 39–42] and references therein). Interaction among identical ratchets can lead to a variety of synchronized dynamics (or collective effects), and stable directed transport can be achieved when the strength of the interaction is larger than the threshold beyond which full synchrony is achieved.

In general, synchronization can be understood as a collective state in which two or more systems, whose dynamics can either be periodic or chaotic, adjust each other giving rise to a common dynamical behaviour [34]. This can be achieved when the oscillators interact, either by coupling or forcing. Synchronization is directly related to the observer problem in control theory in which a feedback mechanism is designed for a receiver (response) system using the transmitted signal of a transmitter (driver) so as to ensure that the controlled receiver synchronizes with the transmitter [35]. On the other hand, the feedback could be such that the information is mutually transmitted among the interacting systems. The design of the effective interaction mechanisms or couplings required to achieve a desired synchronization goal is of current interest. Two fundamental tasks in the analysis and synthesis of such synchronizing systems are the stability of the synchronized state, and a precise determination of the synchronization thresholds—these quantities are particularly relevant from the viewpoint of practical applications [36], because they provide information regarding the operational regime for optimal performance in coupled oscillators.

Recently, we showed that two mutually interacting ratchets in a perturbed asymmetric potential underwent a transition from an on-off intermittently synchronized state to that of full synchronization where collective transport was achieved [37]. In the present paper, we examine the transition to collective current and show that, depending on the strength of the driving, it could be achieved through a chaos-periodic/quasiperiodic transition to full synchronization during which current reversals are completely eliminated, thereby giving rise to fully rectified transport. We also study the stability of the fully synchronized state using the Lyapunov stability theory and linear matrix inequality (LMI) [38], and we then give some sufficient conditions for global asymptotic synchronization, from which we estimate the threshold coupling for the existence of collective and stable controlled transport. The paper is organized as follows: in the next section, we start with a description of coupled ratchets and in section 3 we provide a stability analysis for synchronization. We present numerical results in section 4, and summarize our conclusions in section 5.

2. Coupled inertia ratchets

We consider an archetypal model of two coupled underdamped rocking ratchets [13]. Their dynamics is given by

$$\ddot{x}_i + b\dot{x}_i + \frac{\mathrm{d}V(x_1, x_2)}{\mathrm{d}x_i} = a\cos(\omega t) \qquad (i = 1, 2), \tag{1}$$

where the normalized time t is measured in units of the small resonant frequency ω_0^{-1} of the system, a, ω and b are the amplitude and frequency of the external forcing and the damping parameter, respectively. $V(x_{1,2})$ is the perturbed two-dimensional ratchet potential given as

$$V(x_1, x_2) = 2C - \frac{1}{4\pi^2\delta}[\Phi(x_1) + \Phi(x_2)] + \frac{k}{2}(x_1 - x_2)^2, \tag{2}$$

where $\Phi(x_{1,2}) = \sin 2\pi(x_{1,2} - x_0) + \frac{1}{4}\sin 4\pi(x_{1,2} - x_0)$; the last term is the coupling term, and k is the coupling strength which determines the dynamics and hence the transport properties of equation (1). The parameter x_0 in $\Phi(x_{1,2})$ is chosen such that when $k = 0$ the minima of $V(x_1, x_2)$ are located at the integers, whereas the other parameters are fixed. Here, we use $x_0 = 0.82$, $C = 0.0173$ and $\delta = 1.6$.

Figure 1 shows the perturbed two-dimensional ratchet potential (2) for four different values of the coupling strength k ($k = 0, 0.05, 0.15, 1.0$). The minima and maxima of the potential are marked in blue and red respectively. Note that, as the coupling strength is increased, the maxima of the potential $V(x_1, x_2)$ move outwards, opening up a valley along the diagonal where the two ratchets may most likely share values. This suggests that, for sufficiently large coupling strength, the oscillators would cooperate and achieve optimal transport in some favoured directions.

3. Stability and criteria for controlled transport

The system (1) exhibits intermittent synchronization over a wide range of k values with full synchronization being achieved for large enough coupling strength [37], during which current reversals are fully controlled. The stability of the fully synchronized state was treated in [37] and the exact threshold was obtained numerically. In what follows, we show theoretically that the fully synchronized manifold ($\Delta x(t) = x_1(t) - x_2(t) = 0$) is stable and globally attractive. We establish criteria for global and asymptotic stability of the system (1) on the manifold $\Delta x(t) = 0$, defining the collective state for which current reversals are completely eliminated. We can re-express each isolated ratchet derived from equation (1) as

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -by_i + \sigma\phi(x_i) + a\cos(\omega_0 t) \qquad (i = 1, 2), \end{aligned} \tag{3}$$

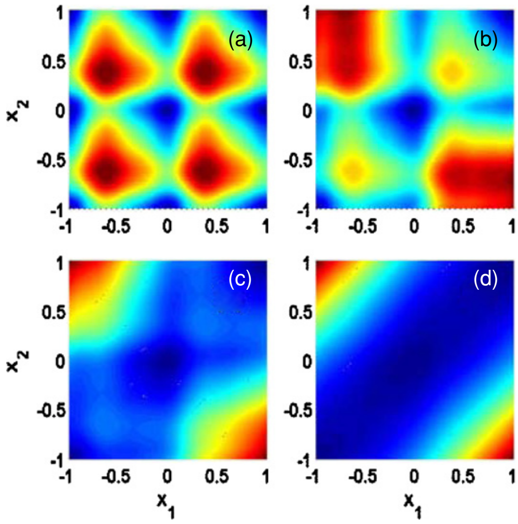

Figure 1. Equipotential contours plot of $V(x_1, x_2)$ with colors growing from blue (minima) to red (maxima): (*a*) no interaction, $k = 0$, (*b*) weak coupling, $k = 0.05$, (*c*) moderate coupling, $k = 0.15$ and (*d*) strong coupling, $k = 1.0$.

where $\phi(x_i) = 2\cos 2\pi(x_i - x_0) + \cos 4\pi(x_i - x_0)$, $\mathbf{x}_i = (x_i, y_i)^T \in \mathbf{R}^2$ $(i = 1, 2)$ are the state variables and $\sigma = \frac{1}{4\pi\delta}$. In compact vector form, the coupled system is

$$\dot{\mathbf{x}}_1 = M\mathbf{x}_1 + \mathbf{f}(\mathbf{x}_1) + \mathbf{m} - \mathbf{u} \tag{4}$$

$$\begin{aligned} \dot{\mathbf{x}}_2 &= M\mathbf{x}_2 + \mathbf{f}(\mathbf{x}_2) + \mathbf{m} + \mathbf{u} \\ \mathbf{u} &= K(\mathbf{x} - \mathbf{y}) \end{aligned} \tag{5}$$

where

$$M = \begin{pmatrix} 0 & 1 \\ 0 & -b \end{pmatrix}, \qquad \mathbf{f}(\mathbf{x}_1) = \begin{pmatrix} 0 \\ \sigma\phi(\mathbf{x}_1) \end{pmatrix},$$

$$\mathbf{f}(\mathbf{x}_2) = \begin{pmatrix} 0 \\ \sigma\phi(\mathbf{x}_2) \end{pmatrix}, \qquad \mathbf{m} = \begin{pmatrix} 0 \\ a\cos\omega_0 t \end{pmatrix},$$

and $\mathbf{K} \in \mathbf{R}^{2\times 2}$ is a constant feedback matrix.

Let us define the synchronization error $\mathbf{e}$, as the difference $\mathbf{x}_1 - \mathbf{x}_2$. Then, by subtracting equation (5) from equation (4) one readily obtains

$$\dot{\mathbf{e}} = (M - 2K + Q)\mathbf{e} \tag{6}$$

where

$$Q = \begin{pmatrix} 0 & 0 \\ q(\mathbf{x}_1, \mathbf{x}_2) & 0 \end{pmatrix},$$

and

$$q(\mathbf{x}_1, \mathbf{x}_2) = \frac{\sigma\phi(x_1, x_2)}{x_1 - x_2}, \tag{7}$$

where $\phi(\mathbf{x}_1, \mathbf{x}_2) = \phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)$. Obviously, $\mathbf{e} = 0$ is an equilibrium point of the error system (6) for vanishing K and full synchronization means that any set of initial conditions satisfies

$$\lim_{t\to\infty} \|\mathbf{e}\| = \lim_{t\to\infty} \|\mathbf{x}_1(\mathbf{t}) - \mathbf{x}_2(\mathbf{t})\| = 0 \tag{8}$$

where $\|\cdot\|$ represents the Euclidean norm of a vector. Thus, we treat the synchronization problem as that of asymptotic stability of the error system (6). For this purpose, we shall employ Lyapunov's stability theory and LMI in [38] to establish criteria for global synchronization according to equation (8). To begin with, we shall apply the following lemma to prove the main theorems of this paper.

Lemma 1. *For $q(x_1, x_2)$ defined by (9), the inequality*

$$|q(x_1, x_2)| \leqslant \frac{2}{\delta} \tag{9}$$

holds.

Proof. Since $x_0 = y_0 =$ constant, by the differential mean-value theorem, the function $\phi(\mathbf{x_1}, \mathbf{x_2})$ can be expressed as

$$\phi(x_1, x_2) = 4\pi(x_1 - x_2)(-\sin\varphi - \sin\eta); \tag{10}$$

where $\varphi, \eta \in [0, 2\pi]$. So that

$$q(\mathbf{x_1}, \mathbf{x_2}) = 4\pi\sigma(-\sin\varphi - \sin\eta) = -\frac{(\sin\varphi + \sin\eta)}{\delta}, \tag{11}$$

and hence inequality (9) holds. □

We proceed by utilizing the stability theory for time-varying systems [38] to derive sufficient criteria for global synchronization in the sense of the error system (6). Here, we propose two theorems. First using the LMI, the following theorem is related to the general control matrix:

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \in \mathbf{R}^{2\times 2}. \tag{12}$$

Theorem 1. *If the coupling matrix K in (12) is chosen such that*

$$\begin{aligned} &4k_{11} + 4k_{22} + 2b > 0 \\ &8k_{11}(2k_{22} + b) > \left(|1 - 2k_{12} - 2k_{21}| + \frac{2}{\delta}\right)^2, \end{aligned} \tag{13}$$

then the coupled systems (4) and (5) achieve full synchrony.

Proof. According to the stability theory of time-varying systems [38], we know that the system (6) is globally asymptotically stable at the equilibrium point $\mathbf{e} = (0, 0)$, if

$$M - 2K + Q(t) + (M - 2K + Q(t))^T = \begin{pmatrix} -4k_{11} & 1 + q - 2(k_{12} + k_{21}) \\ 1 + q - 2(k_{12} + k_{21}) & -2b - 4k_{22} \end{pmatrix} \tag{14}$$

is negative definite. The eigenvalues λ of the matrix (14) above satisfy

$$\lambda^2 + (2b + 4k_{11} + 4k_{22})\lambda + 8k_{11}(b + 2k_{22}) - (1 + q - 2(k_{12} + k_{21}) = 0.$$

According to the Routh–Hurwitz criteria for matrices [44], the matrix (14) is negative definite if and only if

$$\begin{aligned} &2b + 4k_{11} + 4k_{22} > 0, \\ &8k_{11}(b + 2k_{22}) - (1 + q - 2(k_{12} + k_{21}))^2 > 0. \end{aligned} \tag{15}$$

By lemma 1, we have

$$|1 + q - 2k_{12} - 2k_{21}| \leqslant |1 - 2k_{12} - 2k_{21}| + \frac{2}{\delta}. \tag{16}$$

Inequalities (16) hold if conditions (13) are satisfied. This completes the proof. □

Based on lemma 1 and the above theorem some synchronization criteria with respect to the coupling strength may be obtained, which are represented in the following corollaries.

Corollary 1. *If the coupling matrix defined by* $K = \mathrm{diag}(k_1, k_2)$ *is chosen such that*

$$\begin{aligned} &4k_1 + 4k_2 + 2b > 0 \\ &8k_1(2k_2 + b) > \left(1 + \frac{2}{\delta}\right)^2, \end{aligned} \tag{17}$$

then the coupled systems (4) and (5) are fully synchronized.

Proof. Inequalities (17) can be obtained from inequalities (13) by setting $k_{11} = k_1, k_{22} = k_2$ and $k_{12} = k_{21} = 0$. □

Corollary 2. *If the coupling k in equation (1) is selected such that*

$$k > \frac{-b + \sqrt{b^2 + \left(1 + \frac{2}{\delta}\right)^2}}{4} = k^1_{th}, \tag{18}$$

then the coupled systems (4) and (5) are fully synchronized.

Proof. Inequalities (18) can be obtained according to inequalities (17) by setting $k_1 = k_2 = k$. □

Using the Lyapunov stability theory, the following theorem is related to the general control matrix:

$$\mathbf{K} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \in \mathbf{R}^{2\times 2}. \tag{19}$$

Theorem 2. *If there exists a positive definite symmetric matrix given by*

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} > 0 \in \mathbf{R}^2,$$

and the coupling matrix in (19) is chosen such that

$$\begin{aligned} &\Omega_1 = -2k_{11}p_{11} - 2k_{21}|p_{12}| + |p_{12}|\left(\frac{2}{\sigma}\right) < 0 \\ &\Omega_2 = p_{12}(1 - 2k_{12}) - p_{22}(2k_{22} + b) < 0 \\ &4\Omega_1\Omega_2 > \left[p_{11}(1 - 2k_{12}) - p_{12}(2k_{11} + 2k_{22} + b) - 2p_{22}k_{21} + p_{22}\left(\frac{2}{\delta}\right)\right]^2, \end{aligned} \tag{20}$$

then the coupled systems (4) and (5) achieve full synchrony.

Proof. Let us assume a positive definite, decrescent and radially unbounded quadratic Lyapunov function of the form

$$V(e) = \mathbf{e}^T\, \mathbf{Pe}, \tag{21}$$

where $\mathbf{P}$ is a positive definite symmetric matrix as defined earlier. The derivative of the Lyapunov function with respect to time, t, along the trajectory of the error system (6) is of the form

$$\dot{V}(e) = \dot{\mathbf{e}}^T\, \mathbf{Pe} + \mathbf{e}^T\, \mathbf{P}\dot{\mathbf{e}}. \tag{22}$$

Substituting (6) into the system (22), we have

$$\dot{V}(e) = \mathbf{e}^T[(\mathbf{M} - 2\mathbf{K} + \mathbf{Q})^T\mathbf{P} + \mathbf{P}(\mathbf{M} - 2\mathbf{K} + \mathbf{Q})]\mathbf{e} \tag{23}$$

$\dot{V}(e) < 0$ if

$$\gamma = (\mathbf{M} - 2\mathbf{K} + \mathbf{Q})^T\mathbf{P} + \mathbf{P}(\mathbf{M} - 2\mathbf{K} + \mathbf{Q}) < 0, \tag{24}$$

that is

$$\gamma = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{pmatrix}, \tag{25}$$

where $\mu_{11} = -4p_{11}k_{11} + 2p_{12}(q - 2k_{21})$, $\mu_{12} = p_{11}(1 - 2k_{12}) - p_{12}(2k_{11} + 2k_{22} + b) + p_{22}(q - 2k_{21})$ and $\mu_{22} = 2p_{12}(1 - 2k_{12}) - 2p_{22}(b + 2k_{22})$. The above symmetric matrix is negative definite if and only if

$$\begin{aligned} &-4p_{11}k_{11} + 2p_{12}(q - 2k_{21}) < 0 \\ &2p_{12}(1 - 2k_{12}) - 2p_{22}(b + 2k_{22}) < 0 \\ &4[p_{12}(q - 2k_{21}) - 2p_{11}k_{11}][p_{12}(1 - 2k_{12}) - P^* > 0, \end{aligned} \tag{26}$$

where $P^* = [p_{22}(b + 2k_{22})] - [p_{11}(1 - 2k_{12}) - p_{12}(2k_{11} + 2k_{22} + b) + p_{22}(q - 2k_{21})]^2$.

By lemma 1, we have $-4p_{11}k_{11} + 2p_{12}(q - 2k_{21}) \leqslant -4p_{11}k_{11} - 4p_{12}k_{21} + |2p_{12}|q \leqslant 2\Omega_1\,|p_{11}(1 - 2k_{12}) - p_{12}(2k_{11} + 2k_{22} + b) + p_{22}(q - 2k_{21})| \leqslant |p_{11}(1 - 2k_{12}) - p_{12}(2k_{11} + 2k_{22} + b) - 2p_{22}k_{21}| + p_{22}\left(\frac{2}{\delta}\right)$. Inequalities (26) hold if inequalities (20) are satisfied. This completes the proof. □

Based on lemma 1 and the above theorem, some synchronization criteria with respect to the coupling strength may be obtained, which are represented in the following corollaries.

Corollary 3. *If the coupling matrix defined by* $\mathbf{K} = \mathrm{diag}(k_1, k_2)$ *and the positive definite symmetric matrix* $\mathbf{P}$ *defined earlier is chosen such that*

$$\begin{aligned} &k_1 > \frac{|p_{12}|\left(\frac{2}{\delta}\right)}{2p_{11}} \\ &k_2 > \frac{p_{12} - bp_{22}}{2p_{22}} \\ &\beta_1 > \beta_2, \end{aligned} \tag{27}$$

where $\beta_1 = 4\left[|p_{12}|\left(\frac{2}{\delta}\right) - 2p_{11}k_1\right][p_{12} - p_{22}(2k_2 + b)]$ *and* $\beta_2 = \left[|(p_{11} - p_{12}(2k_1 + 2k_2 + b)| + p_{22}\left(\frac{2}{\delta}\right)\right]^2$, *then the coupled systems (4) and (5) achieve full synchrony.*

Proof. Inequalities (27) can be obtained according to inequalities (26) with $k_{11} = k_1, k_{22} = k_2$ and $k_{12} = k_{21} = 0$. □

Corollary 4. *The coupled systems (4) and (5) achieve global synchronization if the coupling matrix* $\mathbf{K} = \mathrm{diag}\{k, k\}$ *and the positive symmetric matrix* $\mathbf{P}$ *defined earlier are chosen such that*

$$\begin{aligned} &k = \max\left(\frac{|p_{12}|\left(\frac{2}{\delta}\right)}{2p_{11}}, \frac{p_{12} - bp_{22}}{2p_{22}}\right) \geqslant 0 \\ &16\left(p_{11}p_{22} - p_{12}^2\right)k^2 - 8k[2p_{22}|p_{12}|\left(\frac{2}{\delta}\right) + P_{b\delta} > 0, \end{aligned} \tag{28}$$

where $P_{b\delta} = p_{11}(p_{12} - bp_{22}) - |p_{12}(p_{11} - bp_{12})|] + 4|p_{12}|\left(\frac{2}{\delta}\right)(p_{12} - bp_{22}) - \left[|p_{11} - bp_{12}| + p_{22}\left(\frac{2}{\delta}\right)\right]^2$.

Proof. Letting $k_1 = k_2 = k$ in the partial synchronization conditions (27), inequality (28) can be obtained.

For $k > 0$ given by (28), we have $\left[p_{11} - p_{12}(4k+b) + p_{22}\left(\frac{2}{\delta}\right)\right]^2 \leqslant \left[|p_{11} - bp_{12}| + 4k|p_{12}| + p_{22}\left(\frac{2}{\delta}\right)\right]^2$. Hence, inequality (28) can be obtained from the partial synchronization condition (27) with $k_1 = k_2 = k$. Since $p_{11}p_{22} - p_{12}^2 > 0$, the solution k to inequality (28) exists. □

Remark. We may select $p_{12} = 0$, $p_{11} = p_{22}\left(\frac{2}{\delta}\right)$, to construct a positive definite matrix

$$\mathbf{P} = p_{22}\begin{pmatrix} \frac{2}{\delta} & 0 \\ 0 & 1 \end{pmatrix}.$$

Based on this matrix, the following algebraic synchronization condition can be obtained from inequalities (28):

$$k > \frac{\sqrt{b^2 + \frac{8}{\delta}} - b}{4} = k_{\text{th}}^2. \tag{29}$$

Note that conditions (18) and (29) are independent of the parameters of the driving force. Thus we expect that, for different choices of external driving, different scenarios would arise. By using parameter values $b = 0.1$ and $\delta = 1.6$, we see that the two criteria yield $k_{\text{th}}^1 = 0.538$ and $k_{\text{th}}^2 = 0.535$, respectively, which are in good agreement.

4. Results and discussion

In this section, we present numerical simulation results to confirm the above analysis. In figure 2, we use three indicators to quantify the transition to collective states, namely (i) the bifurcation diagram for the error states $\mathbf{e}$ defining the difference between the state variables x_1 and x_2 and the velocity of the particle; (ii) the average bare energies h [43] illustrating the interaction mechanism; and (iii) the current J, quantifying the transport. We remark here that our system is highly chaotic and, as such, a single trajectory approach is insufficient to capture its full dynamics; implying that $\mathbf{e}$, h and J have to be averaged out over a large number of trajectories generated from the entire space $[-1, 1] \times [-1, 1]$ which forms the unit cell of the resulting periodic structure. For a long time dynamics T, we can evaluate the error state (in the Poincaré section) for a given trajectory as

$$\mathrm{e}_j = \frac{1}{T}\int_0^T \left(x_1^{(j)} - x_2^{(j)}\right) \mathrm{d}t, \tag{30}$$

where the full error $\mathbf{e} = N^{-1}\sum_{j=1}^{N} \mathbf{e}_j$, is evaluated over the total number of trajectories N. The average bare energies [43] defined as

$$h_{1,2}^{(j)} = \frac{1}{T}\int_0^T h_{1,2}^{(j)}(t)\,\mathrm{d}t; \qquad E_{1,2}^{(j)}(t) = \frac{p_{1,2}^{2(j)}}{2} + V\left(x_1^{(j)}, x_2^{(j)}\right), \tag{31}$$

where $p_{1,2}^{(j)}$ is the associated momentum and $V\left(x_1^{(j)}, x_2^{(j)}\right)$ is the potential, are computed in the same manner. The current of a particle ($i = 1, 2$) over the total number N of trajectories is defined as

$$J_i = \frac{1}{M - n_c}\frac{1}{N}\sum_{l=1}^{M}\sum_{j=1}^{N} \dot{x}_i^{(j)}(t_l) \qquad (i = 1, 2), \tag{32}$$

where t_l of $x_i(t_l)$ is a given observation time. This gives the average velocity, which is then further time-averaged over the number of observations M, where n_c is an empirically obtained cut-off allowing for transient effects, such that a converged current is obtained [16, 37].

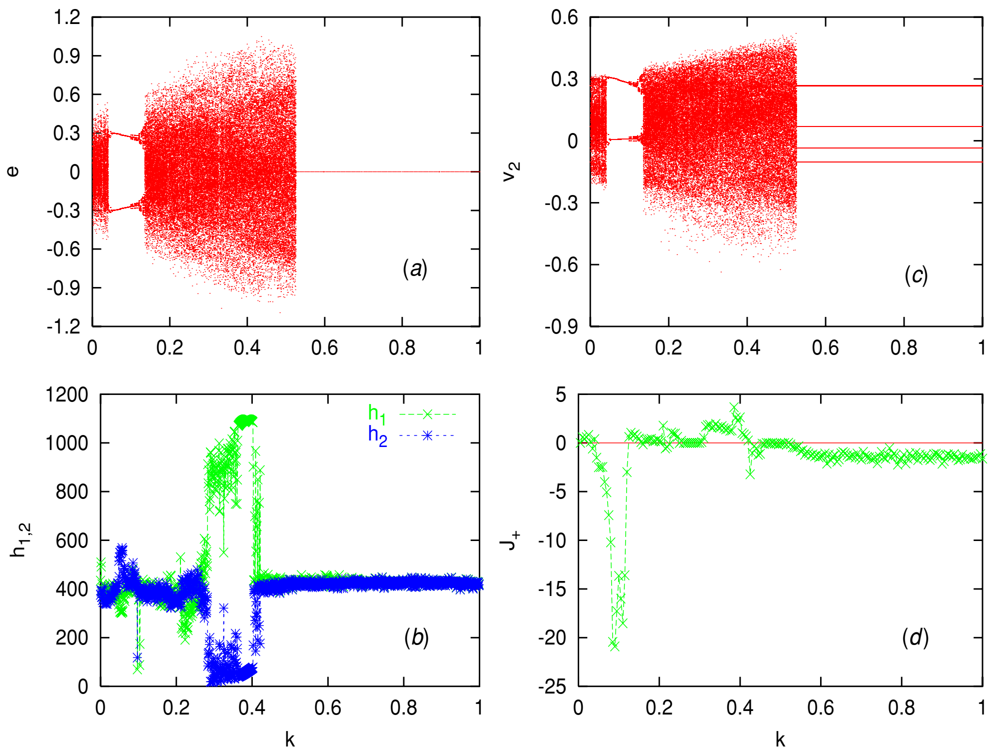

Figure 2. Transition to collective dynamics. (*a*) The bifurcation diagram for **e** versus k shows oscillator locking at the critical bifurcation point, (*b*) the average bare energies $h_{1,2}$ versus k, (*c*) bifurcation diagram of $V_2(\dot{x}_2)$ versus k and (*d*) corresponding ensemble current J_2 in the same coupling range. The parameters are set as $a = 0.080\,9472$, $b = 0.1$, $x_0 = 0.82$ and $\omega = 0.67$.

First, we fix the parameters $b = 0.1$, $\omega = 0.67$ and $a = 0.080\,9472$. We display in figure 2(*a*) the bifurcation of $\mathbf{e_j}$ versus k. By inspection, we find that two remarkable dynamical transitions are apparent. First, a sudden crisis occurs for low coupling strength in which the chaotic behaviour gives way to a period two (P_2) orbit in the periodic window. The P_2 orbit is then annihilated when the strength of the interaction increases, and chaotic behaviour is again re-established for a wide range of k. Secondly, a sudden bifurcation takes place around a critical value k_{cr} ($k_{cr} \approx 0.576$) at which the dynamics of the two ratchets gets locked in complete synchronization. For $k \geqslant k_{cr}$, the orbits of the two oscillators are periodic, as depicted by the bifurcation plot in figure 2(*c*). This is in reasonable agreement with our theoretical predictions given by equations (18) and (29). The corresponding ensemble current $J_+ = J_1 + J_2$ shown in figure 2(*b*), for the same range of coupling, reveals that current reversals still occur prior to the critical coupling strength (k_{cr}). However, for $k > k_{cr}$, the reversals are completely controlled and stable negative transport is achieved.

Both the transition mechanism and the direction in which the particle's motion is rectified in the synchronized state depend on the parameters of the driving force. For fixed driving frequency ($\omega = 0.67$), we show in figure 3 the behaviour of the current J_+ for several values of the driving amplitude a. Clearly, we see three remarkable properties: (i) for $a < 0.08$, the positive direction is favoured and the particles' motion is rectified in this direction when

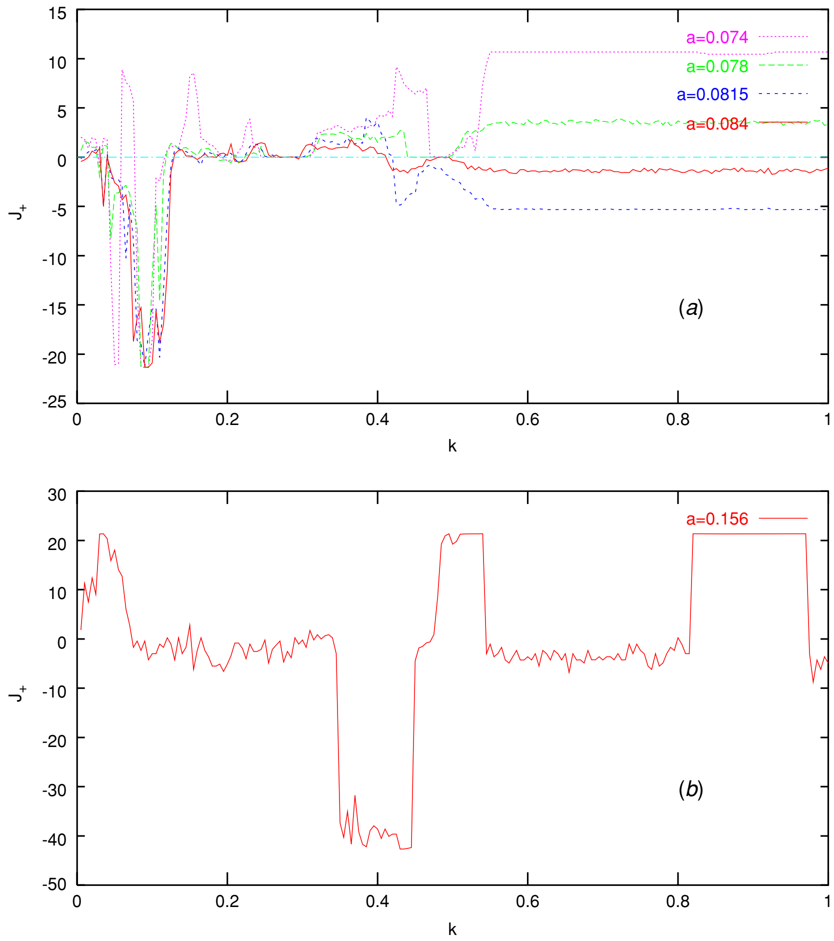

Figure 3. Ensemble current $J_+ = J_1 + J_2$ versus coupling strength k for different driving amplitudes. (a) Weak amplitude $0.07 \leqslant a \leqslant 0.01$, ($b$) strong amplitude $a > 0.1$. Other parameters are fixed as $b = 0.1$, $x_0 = 0.82$ and $\omega = 0.67$.

$k > k_{cr}$ (figure 3(a)); (ii) for $0.08 \leqslant a \leqslant 0.1$, the negative direction is most favourable and the motion is rectified in this direction when $k > k_{cr}$ as shown in figure 3(a)); and (iii) for $a > 0.12$ the direction of rectification is strongly dependent on the value of k (figure 3(b)), at variance with (i) and (ii), showing that conditions (18) and (29) do not hold for large a, typically for $a > 0.12$.

To account for the deviation in (iii), we recall that, for larger values of the driving amplitude a, there is a different bifurcation scenario—namely multistability of attractors are manifest in the uncoupled system [45]. The synchronization dynamics of multistable systems has been a longstanding, outstanding and challenging problem, and the analysis and synthesis is moreover complicated when two identical multistable systems with fractal basin boundaries are coupled, like the system that we study here. Some recent studies have shown that a variety of synchronization behaviours could be observed [50–53], so that the departure which we observed here should be expected. However, a detailed investigation of the characteristics of the collective dynamics is on-going and will be reported elsewhere.

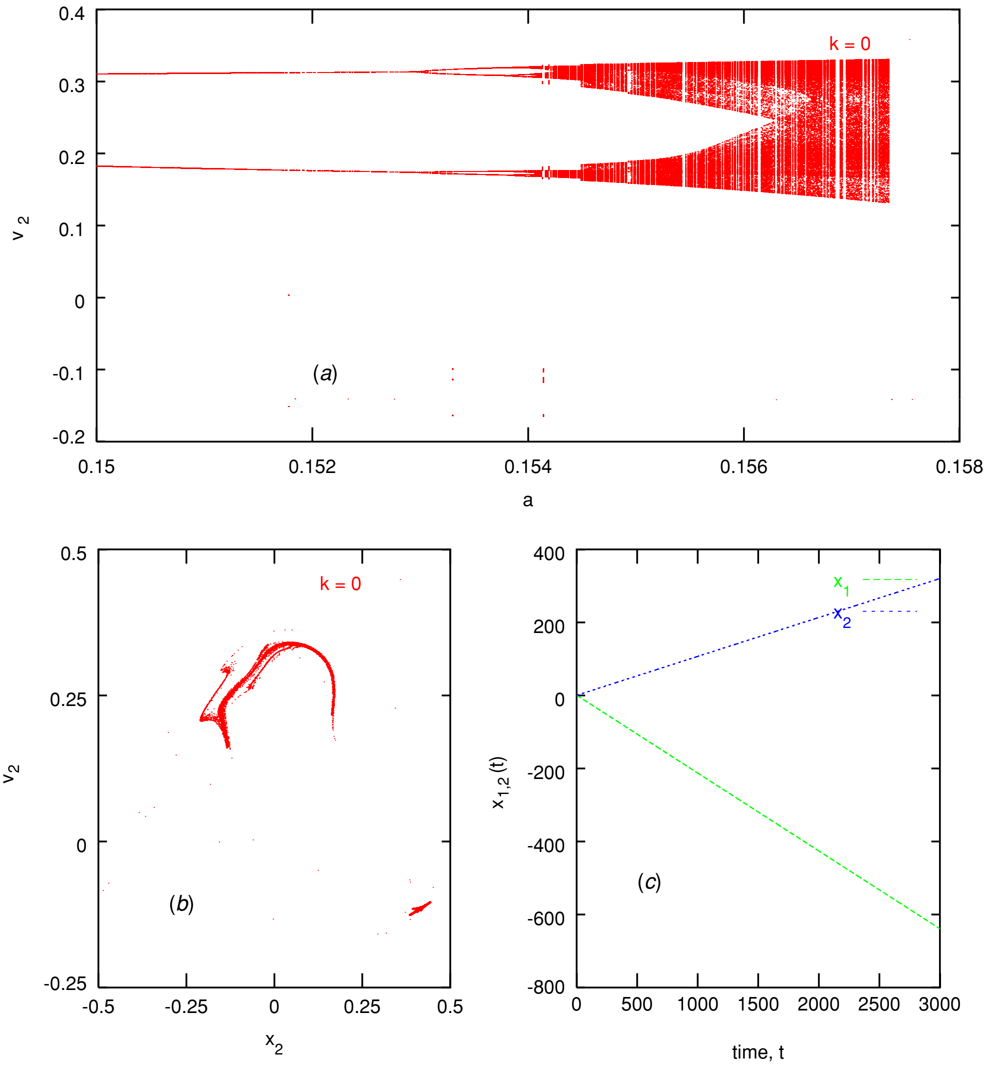

Figure 4. Dynamics of the system (1) when $k = 0$. (*a*) Bifurcation diagram showing the range for co-existing attractors, (*b*) Poincaré plot showing two co-existing attractors for $a = 0.156$, and (*c*) the corresponding trajectories of the attractors in (*b*) obtained using the initial conditions: $(x_0, \dot{x}_0) = (-0.10, 0.25)$ and $(x_0, \dot{x}_0) = (0.43, -0.12)$ (see details in [45]). The other parameters are fixed at $b = 0.1$, $x_0 = 0.82$ and $\omega = 0.67$.

The bifurcation diagram of the system (1) for the driving interval $0.1517 \leqslant a \leqslant 0.1574$ typically shows that chaotic regions co-exist with periodic regions in the interval $0.154 \leqslant a \leqslant 1.574$ [45] (figure 4(*a*)). For $a < 0.154$, a period-1 attractor co-exists with a period-2 attractor. In figure 4(*b*), we show the co-existing attractors for $a = 0.156$ in the Poincaré section, together with the trajectories plotted by using the initial conditions $(x_0, \dot{x}_0) = (-0.10, 0.25)$ and $(x_0, \dot{x}_0) = (0.43, -0.12)$ in figure 4(*c*). Accordingly, the bistable states exemplified above could be interpreted as binary mixtures of particles [23]. While the former case illustrates a binary mixture of non-identical particles of different sizes (i.e. chaotic and periodic), the latter corresponds to a binary mixture of two identical particles (i.e. two periodic orbits). Notably, these situations are very significant and have been observed in recent experiments on transport of K and Rb ions in an ion channel [46], particles of different size in asymmetric silicon pores [49], pinned and interstitial vortices [47] and two different types of vortices [48]. For such co-existing states, the net effects on the directed motion of particles when they interact, and the design of effective control mechanisms aimed at regulating or rectifying the net transport, are challenges that have attracted much attention from researchers. In this direction, Savel'ev *et al* [23] proposed the auxiliary system approach which could be applicable in many practical situations.

Note that the particle transport as captured by figure 3(*b*) shows that, as the strength of the interaction is increased, there is an interplay between the co-existing states such that either state is probable and one of the states (the most probable or stable attractor) would survive and 'drag' the unstable state (attractor) for a given set of driving parameters. Thus, independent of the initial conditions, transport can be achieved in either direction. Specifically, figure 3(*b*) shows that for $0.075 \leqslant k \leqslant 0.475$ and $0.55 \leqslant k \leqslant 0.815$, the negative direction is most probable; thus the particles' motion is rectified in the negative direction; and for $0.47 \leqslant k \leqslant 0.55$ and $0.81 \leqslant k \leqslant 1.0$, the positive direction is the most probable, so that the current direction is positive. It thus clearly shows that the introduction of a specific interaction mechanism provides an efficient means for controlling transport and, in particular, current reversals in non-equilibrium dynamical systems. This has potential applications for the design and operation of high performance and dependable rectifiers, such as arrays of Josephson junction [6], long Josephson junctions [7], asymmetric super conducting quantum interference devices [8] and quantum electronic devices [5].

5. Concluding remarks

In this paper, we have examined two underdamped ratchets coupled via a perturbed asymmetric potential. We have shown that stable transport can occur in their synchronized state, which is achieved through a chaos-quasiperiodic bifurcation transition wherein current reversal is completely eliminated. Based on the Lypunov stability theory and linear matrix inequalities, some necessary and sufficient criteria for stable transport were deduced and an exact analytic estimate of the threshold (k_{th}) for the occurrence of collective transport was obtained. The criteria are expressed in algebraic form. They are strongly dependent on the driving force parameters and valid in the monostable states of the system. In the multistable state where attractors co-exist, the dynamics and transport properties are quite complicated; and in this regime, complete synchronization could not be reached. This requires further investigation and will be reported elsewhere. Finally, we remark that the interaction mechanism employed here could be realized experimentally by linking, for instance, two Josephson junctions in a parallel configuration via ac driving, and, by adjusting the flux, such a device could serve as a voltage rectifier which could be exploited in rapid single flux quantum technology.

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